NON-LOCAL CONSTITUTIVE MODELLING BY THE BOUNDARY ELEMENT METHOD

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Abstract. *The conventional (local) constitutive modelling of materials exhibiting strain softening behaviour is suceptive to a spurious mesh dependence caused by numerically induced strain localization. Also, for very refined meshes, numerical instabilities may be verified, mainly if the simulations are performed by the boundary element method. An alternative to overcome such difficulties is the adoption of the so called non-local constitutive models. In this approach, some internal variables of the constitutive model in a single point are averaged considering its values of the neighbouring points. In this paper, the implicit formulation of the boundary element method for physically non linear problems is used with non-local isotropic damage models. It is shown that the analysis become more stable in comparison to the case of a local application of the same models and that the results recover the desired objectiveness to mesh refinement.*

Keywords: *Non-local constitutive models, Boundary element method, Strain localization*

1 INTRODUCTION

Some material behaviours, e.g., the brittle or quasi-brittle ones, require the introduction of strain softening laws for a correct representation. However, the presence of such laws frequently leads to the development of narrow zones with high displacement gradients (the strain localization bands) while the outside points may experience unloading. Since these zones tend to an infinitely small localization bandwidth with zero energy dissipation, a loss of objectiveness with respect to mesh refinements is verified in numerical analyses. Moreover, in many cases, numerical instabilities are also observed. An alternative to overcome such problems is the adoption of the so called non-local constitutive models. In this approach, some internal variables of the constitutive model in a single point are averaged considering its values of the neighbouring points.

The works of Swedlow and Cruse (1971), Mendelson (1973), Riccardella (1973) and Mukherjee (1977) may be considered the first developments of boundary integral equations for the treatment of physically non-linear materials. Later, Bui (1978) presented a corrected way to evaluate derivatives of the singular integrals involving the inelastic fields, introducing new free terms. Those results were then applied to the boundary element method (BEM) by Telles and Brebbia (1979). They used an explicit formulation, where the increment of the initial fields explicitly appears in the non-linear discretized equations and the equilibrium is achieved by a recursive procedure. Considering the proportional relationship between rates of stress and elastic strain, Telles and Carrer (1991) proposed an implicit formulation, where the initial field increments are written in terms of total strain, resulting in an equilibrium matrix equation, which is linearised and incrementally solved. In all above references, elastoplastic constitutive models were considered.

One way to consider softening behaviour is to adopt a plasticity model with yield limit degradation (reduction), as done by Lin et al. (2002) and Sládek et al. (2003). To aviod the mentioned difficuties when a standard local approach is used, in both works, a non-local procedure, based on the spatial averaging of the plastic multiplier, was also introduced. They differ mainly by the boundary stress evaluation: in the first work, constant functions are used to approximate initial fields inside cells, while, in the second, a regularization of the hypersingular integral equation is adopted. Another non-local plasticity model, based on a re-definition of the yield surface, including a dependency on the Laplacian of the plastic multiplier, was applied to the implicit BEM formulation by Benallal et al. (2002). In that work, a complementary integral representation of the plastic multiplier was discretized and solved together (in a coupled way) to the incremental implicit equation.

An alternative (and perhaps more elegant) way to deal with material's loss of strength, is the adoption of constitutive models based on continuum damage mechanics (CDM). In the BEM context, some works can be cited, such as Rajgelj et al. (1992), Herding and Kuhn (1996), García et al. (1999), Botta et al. (2005) and Peixoto et al. (2016, 2017). In Peixoto et al. (2016), particularly, examples of numerical instabilities were reported. Botta et al. (2005) employed the damage model presented by Comi and Perego (2001). As expected, they also reported mesh dependence when this model is locally applied and an averaging procedure of some strain invariants was introduced, regularizing the model.

In the present work, using the implicit formulation of the BEM, described in Sect. 2, an isotropic damage constitutive model for material representation is adopted in its both standard local and a non-local approach, as presented in Sect. 3. In the last one, the averaged parameter is the equivalent strain used to evaluate the models' historical strain-like internal variable. A very simple numerical procedure, described in Sect. 4, was considered in this averaging process. A representative numerical example is analysed in Sect. 5.

2 IMPLICIT FORMULATION OF THE BEM FOR NON LINEAR MECHANICS

In quasi-static mechanics of continuous media with small displacements, any point in a non-linear equilibrium path can be defined, independently of the constitutive model, as a combination of two linear parts, i.e.,

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^o_{ij} \tag{1}$$

$$\dot{\sigma}_{ij}(\dot{\epsilon}_{ij}) = E^o_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^o_{kl}) = E^o_{ijkl}\dot{\epsilon}_{kl} - \dot{\sigma}^o_{ij} \tag{2}$$

where $\dot{\epsilon}_{ij}$ represents the (rate of) linear strain tensor and $\dot{\epsilon}^e_{ij}$ and $\dot{\epsilon}^o_{ij}$ are its respective elastic and inelastic parts. The fourth-order tensor E^o_{ijkl} is the linear elastic constitutive tensor, while $\dot{\sigma}_{ij}(\dot{\epsilon}_{ij})$ is the Cauchy stress rate, obtained from the actual strain rate, for a given constitutive model.

Thus, the non-linear integral equations in the BEM context are deduced considering the existence of an initial strain field ϵ^o_{ij}, or a corresponding initial stress field $\sigma^o_{ij} = E^o_{ijkl}\epsilon^o_{kl}$. For the first option, such integral equations are presented next, followed by its discrete counterpart and the matrices manipulations to obtain the typical equilibrium vector of the implicit BEM formulation.

2.1 Integral equations

Firstly, the Somigliana's identity for displacements at internal points can be written as

$$\begin{aligned} \dot{u}_i(\boldsymbol{\xi}) = & \int_\Gamma u^*_{ij}(\boldsymbol{\xi}, \mathbf{X})\dot{t}_j(\mathbf{X})d\Gamma - \int_\Gamma t^*_{ij}(\boldsymbol{\xi}, \mathbf{X})\dot{u}_j(\mathbf{X})d\Gamma \\ & + \int_\Omega u^*_{ij}(\boldsymbol{\xi}, \mathbf{X})\dot{b}_j(\mathbf{X})d\Omega + \int_\Omega \sigma^*_{ijk}(\boldsymbol{\xi}, \mathbf{X})\dot{\epsilon}^o_{jk}(\mathbf{X})d\Omega \end{aligned} \tag{3}$$

where u_j and t_j represents the displacement and the traction fields at the boundary Γ and b_j are body forces in the domain Ω. The terms u^*_{ij}, t^*_{ij} and σ^*_{ijk} are Kelvin's fundamental solutions, representing respectively, at a field point $\mathbf{X}$, displacements and tractions in direction j and stress components jk due to a unit concentrated force applied at the collocation point $\boldsymbol{\xi}$ acting in direction i. In Eq. (3), the collocation point is assumed to be internal, i.e., $\{\boldsymbol{\xi} \in \Omega$ and $\boldsymbol{\xi} \notin \Gamma\}$. Expressions for these fundamental solutions are presented in many literature texts, such as Telles (1983); Aliabadi (2002); Gao and Davies (2002).

If the collocation point is located on the domain's boundary, the fundamental solutions' second-order tensors introduce a weakly and a strongly singular character, respectively for the first and second integrals in Eq. (3). Thus, the correct evaluation of the boundary displacement integral equation requires, in this case, a limit process considering a radius of exclusion around the singular point, leading to the following expression:

$$\begin{aligned} c_{ij}(\boldsymbol{\xi})\dot{u}_j(\boldsymbol{\xi}) &= \int_\Gamma u^*_{ij}(\boldsymbol{\xi},\mathbf{X})\dot{t}_j(\mathbf{X})d\Gamma - \fint_\Gamma t^*_{ij}(\boldsymbol{\xi},\mathbf{X})\dot{u}_j(\mathbf{X})d\Gamma \\ &+ \int_\Omega u^*_{ij}(\boldsymbol{\xi},\mathbf{X})\dot{b}_j(\mathbf{X})d\Omega + \int_\Omega \sigma^*_{ijk}(\boldsymbol{\xi},\mathbf{X})\dot{\epsilon}^o_{jk}(\mathbf{X})d\Omega \end{aligned} \tag{4}$$

where $c_{ij}(\boldsymbol{\xi})$ is a function of the boundary's geometry around the collocation point and the material's elastic properties and the second integral in the right side exists only in the Cauchy's Principal Value (CPV) sense, as indicated by the crossed integral symbol.

Furthermore, internal strains can be obtained by taking the symmetric part of the gradient of Eq. (3), related to the collocation point, resulting in

$$\begin{aligned} \dot{\epsilon}_{ij}(\boldsymbol{\xi}) &= \int_\Gamma u^*_{ijk}(\boldsymbol{\xi},\mathbf{X})\dot{t}_k(\mathbf{X})d\Gamma - \int_\Gamma t^*_{ijk}(\boldsymbol{\xi},\mathbf{X})\dot{u}_k(\mathbf{X})d\Gamma \\ &+ \int_\Omega u^*_{ijk}(\boldsymbol{\xi},\mathbf{X})\dot{b}_k(\mathbf{X})d\Omega + \fint_\Omega \sigma^*_{ijkl}(\boldsymbol{\xi},\mathbf{X})\dot{\epsilon}^o_{kl}(\mathbf{X})d\Omega + F^{\epsilon\epsilon}_{ijkl}\dot{\epsilon}^o_{kl}(\boldsymbol{\xi}) \end{aligned} \tag{5}$$

where the last domain integral have a strongly singular kernel when the collocation and field points coincide and, again, its evaluation exists only in the CPV sense. Tensors u^*_{ijk}, t^*_{ijk} and σ^*_{ijkl} are respectively obtained by taking the gradients of fundamental solutions u^*_{ij}, t^*_{ij} and σ^*_{ijk}, while $F^{\epsilon\epsilon}_{ijkl}$ is the free term, which existence was firstly verified by Bui (1978).

2.2 Discrete equations and the implicit formulation

For simplicity, absence of body forces will be considered from now on. Taking the discretization of the problem's boundary (by boundary elements) and the part of the domain where initial fields are present (by cells), as usual in the BEM, Eq. (4) assumes the following form after its application to all boundary collocation points and re-arrangement of boundary prescribed conditions in the same vector:

$$[A]\{\dot{x}\} = [B]\{\dot{y}\} + [Q_{\epsilon^o}]\{\dot{\epsilon}^o\} \tag{6}$$

where $\{\dot{x}\}$ and $\{\dot{y}\}$ are, respectively, the boundary unknowns and the prescribed (rates of) displacement or traction components, $\{\dot{\epsilon}^o\}$ represents the initial strain components and matrices $[A]$, $[B]$ and $[Q_{\epsilon^o}]$ hold the integration coefficients.

Also, by applying Eq. (5) to all internal collocation points, a matrix equation can be written as

$$\{\dot{\epsilon}\} = [A^\epsilon]\{\dot{x}\} + [B^\epsilon]\{\dot{y}\} + [Q^\epsilon_{\epsilon^o}]\{\dot{\epsilon}^o\} \tag{7}$$

where $\{\dot{\epsilon}\}$ represents a vector with rates of strain components. Equation (7) is complemented with strain boundary values as described further.

The Implicit Formulation of the BEM begins by solving Eq. (6) for $\{\dot{x}\}$, i.e.,

$$\{\dot{x}\} = [N]\{\dot{y}\} + [M_{\epsilon^o}]\{\dot{\epsilon}^o\} \tag{8}$$

where

$$[N] = [A^{-1}][B], \qquad [M_{\epsilon^o}] = [A^{-1}][Q_{\epsilon^o}] \tag{9}$$

Then, by applying Eq. (8) to Eq. (7), the following result is obtained:

$$\{\dot{\epsilon}\} = [N^{\epsilon}]\{\dot{y}\} + [M^{\epsilon}_{\epsilon^o}]\{\dot{\epsilon}^o\} \tag{10}$$

where

$$[N^{\epsilon}] = [A^{\epsilon}][A^{-1}][B] + [B^{\epsilon}], \qquad [M^{\epsilon}_{\epsilon^o}] = [A^{\epsilon}][A^{-1}][Q_{\epsilon^o}] + [Q^{\epsilon}_{\epsilon^o}] \tag{11}$$

For rate independent constitutive models, the time derivatives may be substituted by finite increments, i.e., $(\dot{\cdot}) = (\cdot)_i - (\cdot)_{i-1}$. Thus, considering the i-th increment of the prescribed loads, $\{y\}$, Eq. (10) is rewritten as

$$\{\epsilon\}^i = \lambda^i[N^{\epsilon}]\{y\} + [M^{\epsilon}_{\epsilon^o}]\{\epsilon^o\}^i \tag{12}$$

where the load factor λ^i is a cumulative scalar quantity which defines the magnitude of the external load increment. Its value is defined by the control method employed, as described in Peixoto et al. (2016).

From Eq. (12), an equilibrium condition vector, in function of strain components and the load factor, $\{Q\}^i \equiv \{Q(\epsilon^i, \lambda^i)\}$, can be defined with the introduction of the matrix form of Eq. (2), i.e.,

$$\{Q\}^i = \lambda^i[N^{\epsilon}]\{y\} + [M^{\epsilon}_{\epsilon^o}]\big(\{\epsilon\}^i - [E^o]^{-1}\{\sigma(\epsilon)\}^i\big) - \{\epsilon\}^i = \{0\} \tag{13}$$

where $[E^o]$ is a quasi-diagonal matrix representing the linear elastic constitutive relationship and $\{\sigma(\epsilon)\}$ is the appropriate stress vector, corresponding to the strain state, in the constitutive model chosen.

The linearisation of Eq. (13) permits its evaluation in each incremental step by Newton's method. For more details about the complete solution strategy, see Peixoto et al. (2017).

Integrals treatment

All coefficients which refers to boundary points of the matrices in Eq. (7) can be evaluated using a hyper-singular integral equation - obtained from the gradient of Eq. (4) - or, as more commonly, by using a local recovery procedure - see, for example, Gao and Davies (2002). This last approach was used in this work.

The regular integrals in Eqs. (4) and (5) are solved by conventional Gaussian quadrature. The boundary and domain weakly singular integrals, involving the kernels containing u^*_{ij} and σ^*_{ijk}, are solved using the methods presented respectively in Huang and Cruse (1993) and Lachat and Watson (1976). For evaluation of the boundary strongly singular integral in Eq. (4), the methodology described in Guiggiani and Casalini (1987) was employed, while for the domain strongly singular integral of Eq. (5), the procedure from Gao and Davies (2000) was adopted.

3 LOCAL AND NON-LOCAL CONSTITUTIVE MODELLING

In this work, the isotropic damage constitutive model proposed by Simo and Ju (1987) was adopted to represent the material behaviour. Such a model can be summarized by the next expressions:

$$\text{Free energy:}\quad \psi(\epsilon_{ij}, r) = [1 - D(r)]\psi_o(\epsilon_{ij}),\quad \psi_o(\epsilon_{ij}) = \frac{1}{2}\epsilon_{ij}E^o_{ijkl}\epsilon_{kl}; \tag{14a}$$

$$\text{Constitutive equation:}\quad \sigma_{ij} = \frac{\partial\psi(\epsilon_{ij}, r)}{\partial\epsilon_{ij}} = (1 - D)E^o_{ijkl}\epsilon_{kl} = E_{ijkl}\epsilon_{kl}; \tag{14b}$$

$$\text{Damage variable:}\quad D \equiv D(r) = 1 - \frac{q(r)}{r},\quad D \in [0, 1]; \tag{14c}$$

$$\text{Internal variable evolution law:}\quad \dot{r} = \dot{\lambda},\quad \begin{cases} r \in [r_o, \infty), \\ r_o = r|_{t=0} = \frac{f_t}{\sqrt{E}}; \end{cases} \tag{14d}$$

$$\text{Damage criterion:}\quad \begin{cases} F(\sigma_{ij}, q) \equiv \tau_\sigma - q = \sqrt{\sigma_{ij}E^{o,-1}_{ijkl}\sigma_{kl}} - q \quad \text{(stress space)} \\ \text{or} \\ \bar{F}(\epsilon_{ij}, r) \equiv \tau_\epsilon - r = \sqrt{\epsilon_{ij}E^o_{ijkl}\epsilon_{kl}} - r \quad \text{(strain space);} \end{cases} \tag{14e}$$

$$\text{Loading-unloading conditions:}\quad F \leqslant 0,\quad \dot{\lambda} \geqslant 0,\quad \dot{\lambda}F = 0,\quad \dot{\lambda}\dot{F} = 0 \tag{14f}$$

$$\text{Hardening rule:}\quad \dot{q} = H(r)\dot{r},\quad (H = q'(r) \leqslant 0),\quad \begin{cases} q \in [0, r_o], \\ q|_{t=0} = r_o \end{cases} \tag{14g}$$

where $E_{ijkl} = (1 - D)E^o_{ijkl}$ is the secant constitutive tensor and Eq. (14b) is used to locally evaluate the stress vector $\{\sigma(\epsilon)\}$, from the actual strain state, in Eq. (13). Parameters q and r are respectively the stress- and strain-like internal variables. The value r_o is the threshold that determines the initial elasticity domain, which can be characterized in terms of the uniaxial strength f_t and the elastic modulus E - see Eq. (14d). The parameter H is called as hardening-softening modulus which evolution is defined after postulation of a law $q(r)$, understood as a material property. The same may be said for the evolution of the damage variable, D (Eq. 14c). In Eq. (14e), the norms τ_σ and τ_ϵ are correspondent in a constitutive point of view and, particularly the second one, is usually known as equivalent strain.

From Eqs. (14b), (14c), (14e) and (14f), a tangent constitutive relation can be written as

$$\dot{\sigma}_{ij} = E^t_{ijkl}\dot{\epsilon}_{kl};\quad E^t_{ijkl} = E_{ijkl} - \left(\frac{\partial D}{\partial r}\right)\left(\frac{\partial\tau_\epsilon}{\partial\epsilon_{kl}}\right)E^o_{ijrs}\epsilon_{rs} \tag{15}$$

where E^t_{ijkl} is called as the constitutive tangent tensor (or operator), and is required in the linearisation process of Eq. (13).

In the standard local constitutive modelling, not only strain and stress, but also all other variables, such as the equivalent strain, are point-to-point evaluated. As mentioned, for stress-strain relations with softening, this procedure leads to pathological sensitivity to spatial discretization, e.g., the size of internal cells and, in many cases, to numerical instabilities.

As an alternative, the non-local constitutive models are based in a averaging scheme of one, or some, variables. In this case, the stress at a certain point become dependent on the distribution of these variables over the whole body's domain or, at least, on their distribution in a finite neighborhood of the point under consideration. A weighting function is commonly adopted in order to guarantee more influence of the closest points.

The choice of the variable to be averaged remains to some extent arbitrary. In this work, we followed the idea of Jirsek (2004) and adopted as the non-local variable the equivalent strain, now given by:

$$\tau_\epsilon(\boldsymbol{\xi}) = \int_{\tilde{\Omega}} \alpha(\boldsymbol{\xi}, \mathbf{X})\tau_\epsilon(\mathbf{X})\, d\Omega \tag{16}$$

where $\tilde{\Omega}$ is the averaging domain and $\alpha(\boldsymbol{\xi}, \mathbf{X})$ is a weighting function that depends on the distance of a field point, $\mathbf{X}$, from the origin point, $\boldsymbol{\xi}$ (the point where the equivalent strain is being evaluated).

Equation (16) is then used for evaluation of a current value of the strain-like internal variable through Eqs. (14d) to (14f), i.e.,

$$r(\boldsymbol{\xi}, t) = \max_{\tau \leqslant t}\left[r_o, \tau_\epsilon(\boldsymbol{\xi}, \tau)\right] \tag{17}$$

where t means the time (or, in a numerical point-of-view, the pseudo-time for time-independent constitutive models, such as the one used here) from the beginning of load application.

4 NUMERICAL AVERAGING OF NON-LOCAL VARIABLES

The examples in this work were restricted to bidimensional problems, however, the extension to tridimensional ones is straightforward.

A very simple numerical treatment was considered for evaluation of the integral in Eq. (16). Basically, as depicted in Fig. 1, for each field collocation point, $\mathbf{X}_k$, was defined an area of influence, Ω_k, which is equal the sum of fractions of the adjacent cells' areas. Such fractions are the total area of a cell divided by its number of collocation points.

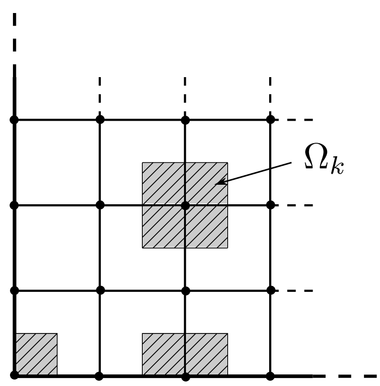

Figure 1: Influence area of a collocation point for non-local averaging procedure

Also, to define the range of $\tilde{\Omega}$, a real parameter β needs to be specified before the analysis.

Thus, the non-local equivalent strain in a collocation point $\boldsymbol{\xi}$ is calculated by:

$$\tau_\epsilon(\boldsymbol{\xi}) = \sum_k \alpha(\boldsymbol{\xi}, \mathbf{X}_k)\tau_\epsilon(\mathbf{X}_k)\Omega_k \tag{18}$$

where only points which distance from $\boldsymbol{\xi}$ is less than β are taken for the summation.

5 NUMERICAL EXAMPLE

As a numerical example, a notched strip under traction, presented in Fig. 2a, was considered. Due to its double symmetry, only a quarter of the problem was modelled (under plane stress assumption), as depicted in Fig. 2b.

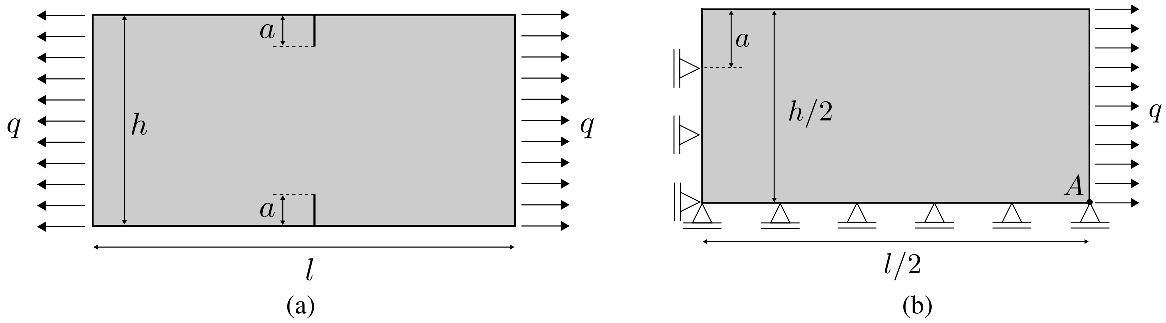

Figure 2: Example - Notched strip under traction: (a) actual problem; (b) numerical model.

The adopted geometrical parameters were $l = 400$ mm, $h = 200$ mm, $a = 33$ mm and a strip width of 50 mm.

For material behaviour, the isotropic damage constitutive model, described in Sect. 3, was used in its local and non-local versions. An exponential relation between the stress- and strain-like internal variables was postulated, such as:

$$q(r) = r_o e^{-b(r-r_o)} \quad \Rightarrow \quad D(r) = 1 - \frac{r_o}{r} e^{-b(r-r_o)} \tag{19}$$

with

$$b = \frac{k f_t}{G_f \sqrt{E}} \tag{20}$$

where k and G_f are, respectively, a size parameter and the material's fracture energy density. Its assumed values were $k = 10$ mm and $G_f = 0.1$ N/mm. The uniaxial tensile strength, the Young's modulus and the Poisson's ratio were considered as $f_t = 2.0$ MPa, $E = 20$ GPa and $\nu = 0.2$.

Particularly for the non-local model analyses, a Gaussian distribution was used as weighting function to take the averaged value of a point's equivalent strain, i.e.,

$$\alpha(\boldsymbol{\xi}, \mathbf{X}) = e^{-(\bar{r}/2\beta)^2} \tag{21}$$

where $\bar{r}$ is the distance between $\mathbf{X}$ and $\boldsymbol{\xi}$ and β is the same parameter that defines the range of averaging domain, described in Sect. 4, here adopted as $\beta = 30$ mm.

The analyses were performed with four different meshes, illustrated in Fig. 3, of linear boundary elements and cells, which total quantities are presented in Table 1.

The non linear incremental-iterative process was controlled by setting a fixed horizontal displacement for point A (see Fig. 2b). Curves for total tensile load in function of such displacement component are presented in Fig. 4 for the standard local constitutive model.

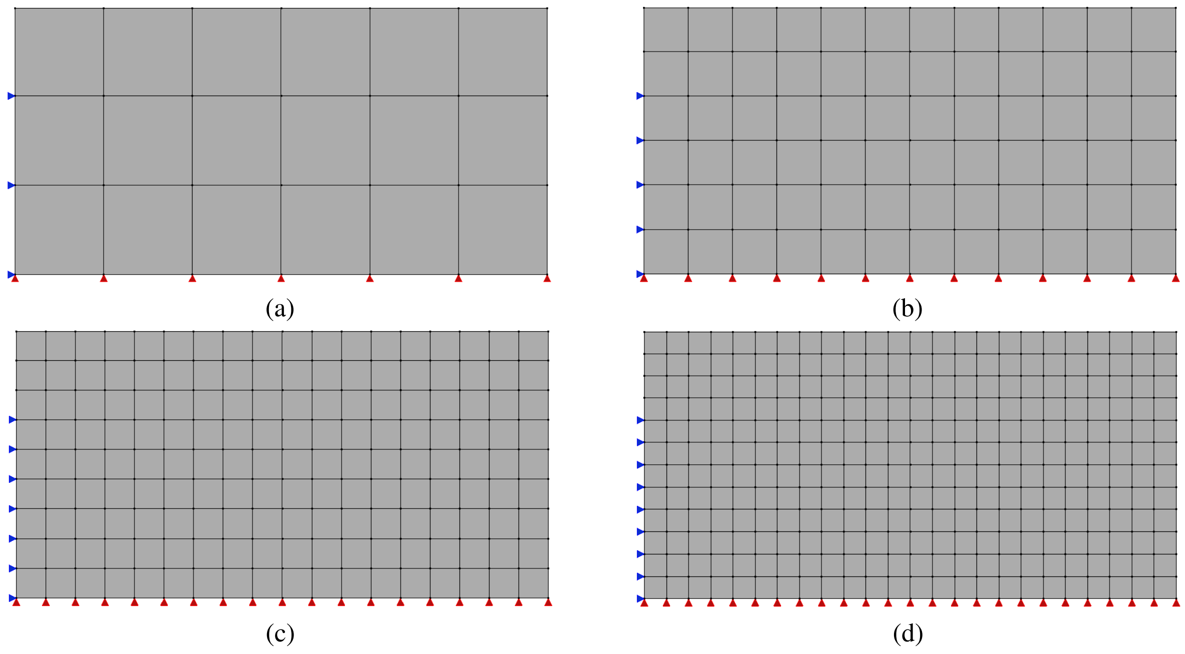

Figure 3: Example - Meshes: (a) mesh 1; (b) mesh 2 ; (c) mesh 3; (d) mesh 4.

Table 1: Meshes sizes

Mesh	Number of B.E.	Number of Cells	Cells' size
Mesh 1	18	18	(33 x 33) mm
Mesh 2	36	72	(17 x 17) mm
Mesh 3	54	162	(11 x 11) mm
Mesh 4	72	288	(8 x 8) mm

In this case, the two more refined meshes shown numerical instabilities and the softening branch was not reproduced. Moreover, a spurious mesh dependence can be easily noted by the responses of the two coarse meshes, in which the maximum load peaks differ considerably.

For the analyses with the non-local constitutive model, such instabilities did not occur, as depicted in Fig. 5, and the equilibrium paths tend to a single curve as the mesh is refined, overcoming its undesired dependence. Its important to emphasize that the analysis with mesh 1 led to the same results as its correspondent with the standard local model since, in this case, the cells edges were longer than parameter β and only the local value of the equivalent strain was used in the averaging process. Also, a very similar peak load was obtained for meshes 2 to 4, as shown in Fig. 6.

Finally, spatial distributions of the damage variable, $D(r)$, are plotted in Fig. 7 for mesh 2 and both, local and non-local approaches, after 50 (a quarter of the analyses) and 200 steps (the end of the analyses). Smoother configurations were verified for the non-local model (Figs. 7c and 7d).

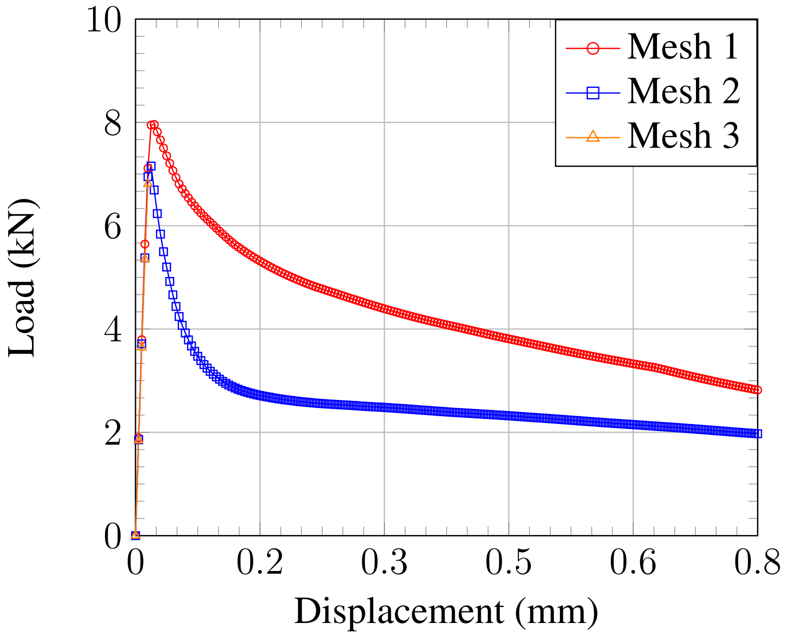

Figure 4: Local constitutive models equilibrium paths: Total load vs. Point A displacement.

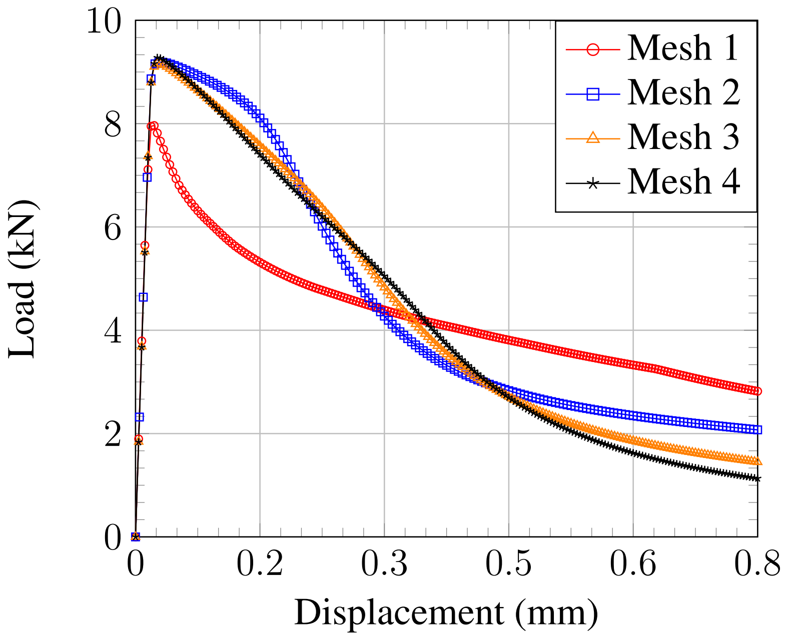

Figure 5: Non-local constitutive models equilibrium paths: Total load vs. Point A displacement.

6 CONCLUDING REMARKS

Its well-known in physically non-linear solid mechanics that constitutive models equipped with a strain softening law leads to spurious mesh sensitivity and, in many cases, to numerical instabilities. By the authors experience, as reported in Peixoto et al. (2016), such instabilities seem to be more common in analyses by the boundary element method than other numerical methods.

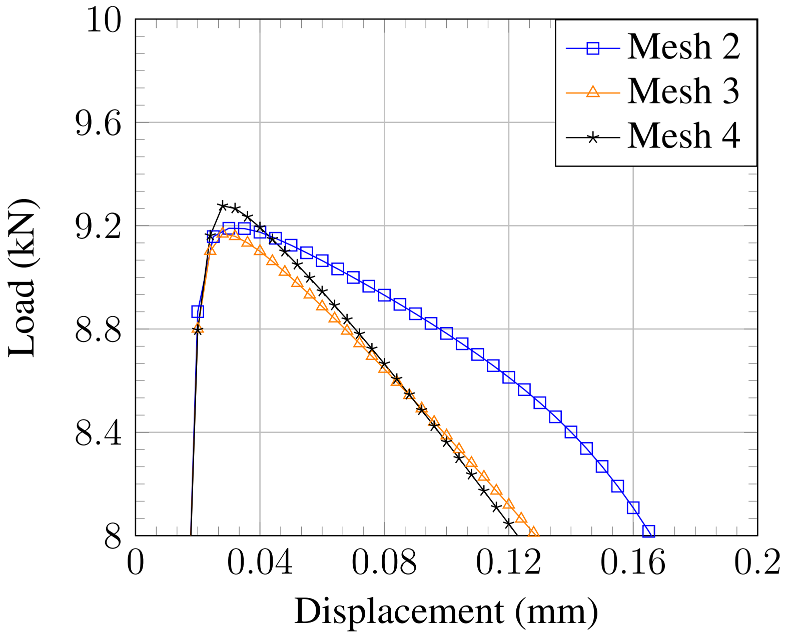

Figure 6: Non-local constitutive models equilibrium paths: peak load.

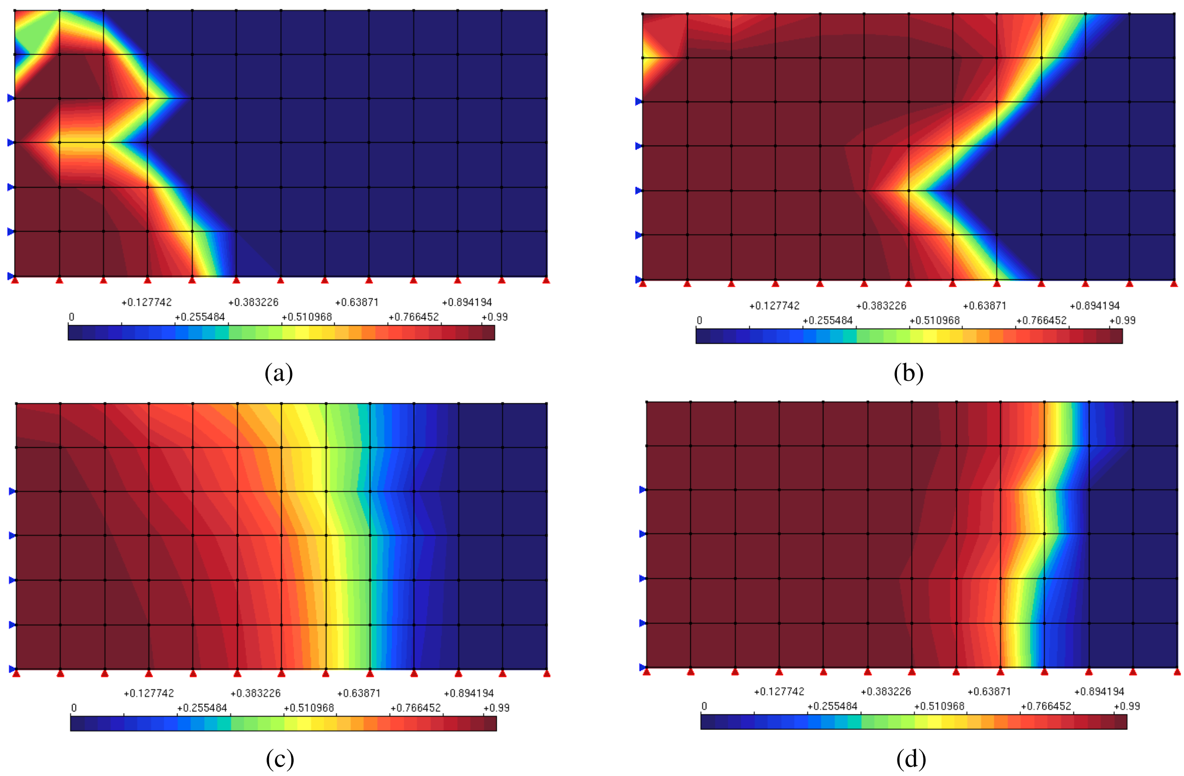

Figure 7: Damage variable distribution: (a) local model - step 50; (b) local model - step 200; (c) non-local model - step 50; (d) non-local model - step 200.

In this work, an isotropic damage constitutive model was used in its standard local approach and also in a non-local version, in which the historical strain-like internal variable in a point is determined not by its local equivalent strain but by an averaged value of such parameter, taking some neighbouring points, weighted by a Gaussian distribution function. A very simple strategy was adopted to average this non-local variable. As described in Sect. 4, an area of influence is defined for each collocation point, given by the sum of a proportional part of its adjacent cells' areas, over which the equivalent strain is considered as constant.

Although using such a simple average scheme, interesting results were obtained for the numerical example presented in Sect. 5. It has been verified that, between the four meshes adopted, only the analyses with the two more coarse progressed after the load peak, which, however, presented remarkably different values (Fig. 4). Also, its softening branches considerably differ from each other indicating numerically induced strain localization, which can be corroborated by the diffuse distributions of the damage variable shown in Figs. 7a and 7b.

When the non-local approach was considered, no numerical instabilities were verified and the equilibrium paths tend to a single curve with discretization refinement (Fig. 5). Also, the load peaks are almost coincident for all meshes in which the non-local parameter was effectively averaged (Fig. 6). In Figs. 7c and 7d, more realistic patterns for the damage variable may be observed.

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REFERENCES

Aliabadi, M. H., 2002. *The boundary element method: volume 2 - applications in solids and structures*. John Wiley & Sons, Chichester.

Benallal, A., Fudoli, C. A., & Venturini, W. S., 2002. An implicit BEM formulation for gradient plasticity and localization phenomena. *International Journal for Numerical Methods in Engineering*, vol. 53, pp. 1853–1869.

Botta, A. S., Venturini, W. S., & Benallal, A., 2005. BEM applied to damage models emphasizing localization and associated regularization techniques. *Engineering Analysis with Boundary Elements*, vol. 29, pp. 814–827.

Bui, H. D., 1978. Some remarks about the formulation of three-dimensional thermoelastoplastic problems by integral equations. *International Journal of Solids and Structures*, vol. 14, pp. 935–939.

Comi, C. & Perego, U., 2001. Fracture energy based bi-dissipative damage model for concrete. *International Journal of Solids and Structures*, vol. 38, pp. 6427–6454.

Gao, X.-W. & Davies, T. G., 2000. An effective boundary element algorithm for 2D and 3D elastoplastic problems. *International Journal of Solids and Structures*, vol. 37, pp. 4987–5008.

Gao, X.-W. & Davies, T. G., 2002. *Boundary Element Programming in Mechanics*. Cambridge University Press, Cambridge.

García, R., Flórez-López, J., & Cerrolaza, M., 1999. A boundary element formulation for a class of non-local damage models. *International Journal of Solids and Structures*, vol. 36, pp. 3617–3638.

Guiggiani, M. & Casalini, P., 1987. Direct computation of Cauchy principal value integrals in advanced boundary elements. *International Journal for Numerical Methods in Engineering*, vol. 24, pp. 1711–1720.

Herding, U. & Kuhn, G., 1996. A field boundary element formulation for damage mechanics. *Engineering Analysis with Boundary Elements*, vol. 18, pp. 137–147.

Huang, Q. & Cruse, T. A., 1993. Some notes on singular integral techniques in boundary element analysis. *International Journal for Numerical Methods in Engineering*, vol. 36, pp. 2643–2659.

Jirsek, M., 2004. Non-local damage mechanics with application to concrete. *Revue franaise de génie civil*, vol. 8, pp. 683–707.

Lachat, J. C. & Watson, J. O., 1976. Effective numerical treatment of boundary integral equations - a formulation for three dimensional elastostatics. *International Journal for Numerical Methods in Engineering*, vol. 10, pp. 991–1005.

Lin, F.-B., Yan, G., Bažant, Z. P., & Ding, F., 2002. Non-local strain softening model of quasi-brittle materials using boundary element method. *Engineering Analysis with Boundary Elements*, vol. 26, pp. 417–424.

Mendelson, A., 1973. Boundary integral methods in elasticity and plasticity. Technical report, NASA TN D-7418, USA.

Mukherjee, S., 1977. Corrected boundary-integral equations in planar thermoelastoplasticity. *International Journal of Solids and Structures*, vol. 13, pp. 331–335.

Peixoto, R. G., Anacleto, F. E. S., Ribeiro, G. O., Pitangueira, R. L. S., & Penna, S. S., 2016. A solution strategy for non-linear implicit BEM formulation using a unified constitutive modelling framework. *Engineering Analysis with Boundary Elements*, vol. 64, pp. 295–310.

Peixoto, R. G., Ribeiro, G. O., Pitangueira, R. L. S., & Penna, S. S., 2017. The Strong Discontinuity Approach as a limit case of strain localization in the implicit BEM formulation. *Engineering Analysis with Boundary Elements*, vol. 80, pp. 127–141.

Rajgelj, S., Amadio, C., & Nappi, A., 1992. Application of damage mechanics concepts to the boundary element method. In Brebbia, C. A. & Ingber, M. S., eds, *Seventh International Conference on Boundary Element Technology*, pp. 617–634.

Riccardella, P. C., 1973. An implementation of the boundary integral technique for planar problems in elasticity and elastoplasticity. Technical report, SM-73-10, Carnegie Mellon University, Pittsburgh, USA.

Simo, J. C. & Ju, J. W., 1987. Strain- and stress-based continuum damage models - I. Formulation. *International Journal of Solids and Structures*, vol. 23, n. 7, pp. 821–840.

Sládek, J., Sládek, V., & Bažant, Z. P., 2003. Non-local boundary integral formulation for softening damage. *International Journal for Numerical Methods in Engineering*, vol. 57, pp. 103–116.

Swedlow, J. L. & Cruse, T. A., 1971. Formulation of boundary integral equations for three dimensional elastoplastic flow. *Journal of Solids and Structures*, vol. 7, pp. 1673–1683.

Telles, J. C. F., 1983. *Boundary Element Method applied to inelastic problems*. Springer-Verlag, Berlin.

Telles, J. C. F. & Brebbia, C. A., 1979. On the application of the boundary element method to plasticity. *Applied Mathematical Modelling*, vol. 3, n. 4, pp. 466–470.

Telles, J. C. F. & Carrer, J. A. M., 1991. Implicit procedures for the solution of elastoplastic problems by the boundary element method. *Mathematical and Computer Modelling*, vol. 15, pp. 303–311.